Classification
& Logistic Regression
& maybe deep learning

Slides by:

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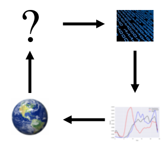
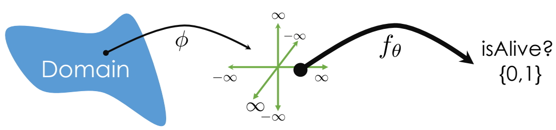
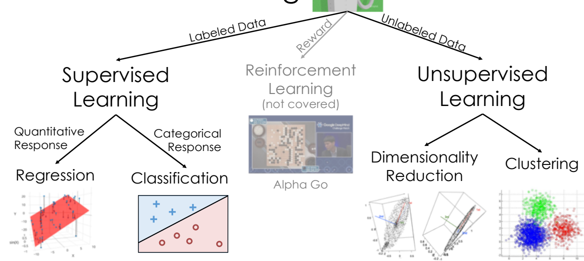


Previously…

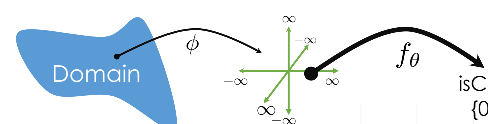
Classification



Taxonomy of Machine Learning

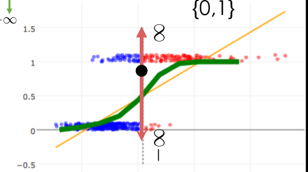


Classification



Can we just use least squares?

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}(y_i - f_\theta(x_i))^2 + \lambda \mathbf{R}(\theta)$$



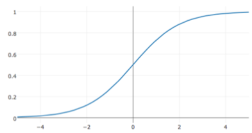
Defining a New Model for Classification

Logistic Regression

- Model the probability of a particular label:

$$\hat{\mathbf{P}}_\theta\left(y = 1 \,|\, x\right) = \sigma(\underbrace{\phi(x)^T\theta}_{\text{Linear Model}}) = \frac{1}{1+\exp\left(-\phi(x)^T\theta\right)}$$

$$\sigma(t) = \frac{1}{1+\exp(-t)}$$



Motivation for the Logistic Model

- Model the "log odds" as a linear model

$$\underbrace{\phi(x_i)^T\theta}_{\text{Linear Model}} = \underbrace{\log\left(\frac{\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}{\hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)}\right)}_{\text{Log odds}}$$

$$\phi(x_i)^T\theta = 0 \overset{\exp(0)=1}{\Rightarrow} \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)$$

$$\phi(x_i)^T\theta > 0 \overset{\exp(\epsilon)>1}{\Rightarrow} \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) > \hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)$$

$$\phi(x_i)^T\theta < 0 \overset{\exp(-\epsilon)<1}{\Rightarrow} \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) < \hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)$$

for any positive ϵ

Motivation for the Logistic Model

$$\phi(x_i)^T\theta = \log\left(\frac{\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}{\hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)}\right)$$

$$= \log\left(\frac{\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}{1-\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}\right)$$

Taking the exponent of both sides

$$\exp\left(\phi(x_i)^T\theta\right) = \frac{\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}{1-\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}$$

$$\exp\left(\phi(x_i)^T\theta\right) = \frac{\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}{1-\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)}$$

Algebra

$$\exp\left(\phi(x_i)^T\theta\right)\left(1-\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)\right) = \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)$$

Expanding terms

$$\exp\left(\phi(x_i)^T\theta\right) - \exp\left(\phi(x_i)^T\theta\right)\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)$$

Collect terms on the other side ...

$$\exp\left(\phi(x_i)^T\theta\right) = \hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)\left(1+\exp\left(\phi(x_i)^T\theta\right)\right)$$

Solving for P(y=1|x)

$$\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \frac{\exp\left(\phi(x_i)^T\theta\right)}{1+\exp\left(\phi(x_i)^T\theta\right)}$$

Solving for P(y=1|x)

$$\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \frac{\exp\left(\phi(x_i)^T\theta\right)}{1+\exp\left(\phi(x_i)^T\theta\right)}$$

Dividing numerator and denominator by $\exp(\phi(x_i)^T\theta)$

$$\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \frac{1}{1+\exp\left(-\phi(x_i)^T\theta\right)}$$

$$= \sigma(\phi(x)^T\theta)$$

Where

$$\sigma(t) = \frac{1}{1+\exp(-t)}$$

The Logistic Regression Model

Model: $\hat{\mathbf{P}}_\theta\left(y = 1 \,|\, x\right) = \sigma(\phi(x)^T\theta) = \frac{1}{1+\exp\left(-\phi(x)^T\theta\right)}$

How do we fit the model to the data?

Defining the Loss

Could we use the Squared Loss

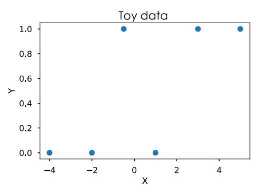
- What about squared loss and the new model:

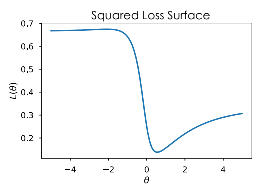
$$L(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sigma\left(\phi(x_i)^T\theta\right)\right)^2$$

  - Tries to match probability with 0/1 labels.
  - Occasionally used in some neural network applications
  - **Non-convex!**

$$L(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sigma\left(\phi(x_i)^T\theta\right)\right)^2$$

- Tries to match probability with 0/1 labels.
- Occasionally used in some neural network applications
- **Non-convex!**





Defining the Cross Entropy Loss

Loss Function

- We want our model to be close to the data:

$$\hat{\mathbf{P}}_\theta(y=1\,|\,x) \approx \mathbf{P}(y=1\,|\,x)$$

- Example: (cute or not)?

x = 

y = 1 "cute"

	Cute	Not Cute
Observed Probability	$\mathbf{P}(y=1\,\|\,x)$ = 1.0	$\mathbf{P}(y=0\,\|\,x)$ = 0.0
Predicted Probability	$\hat{\mathbf{P}}_\theta(y=1\,\|\,x)$ = 0.8	$\hat{\mathbf{P}}_\theta(y=0\,\|\,x)$ = 0.2

Loss Function

- We want our model to be close to the data:

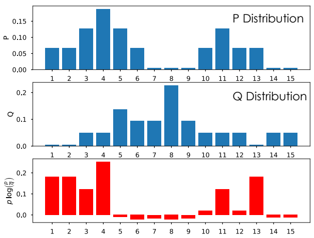
$$\hat{\mathbf{P}}_\theta(y=1\,|\,x) \approx \mathbf{P}(y=1\,|\,x)$$

- Kullback–Leibler (KL) Divergence provides a measure of difference between two distributions:
  - Difference between two discrete distributions P and Q

$$\mathbf{D}(P||Q) = \sum_{k=0}^{K-1} P(k)\log\left(\frac{P(k)}{Q(k)}\right)$$

Kullback–Leibler (KL) Divergence

$$\mathbf{D}(P||Q) = \sum_{k=0}^{K-1} P(k) \log\left(\frac{P(k)}{Q(k)}\right)$$



- The average log difference between P and Q weighted by P
- Does not penalize mismatch for rare events with respect to P
- Note that it is not symmetric

$$\mathbf{D}(P||Q) \neq \mathbf{D}(Q||P)$$

Loss Function

- We want our model to be close to the data:

$$\hat{\mathbf{P}}_\theta\left(y = 1 \,|\, x\right) \;\approx\; \mathbf{P}\left(y = 1 \,|\, x\right)$$

- Kullback–Leibler (KL) divergence for classification
  - For a **single** (x,y) data point

=2 Binary Classification

$$\mathbf{D}_{KL}\left(\mathbf{P}||\hat{\mathbf{P}}_\theta\right) = \sum_{k=0}^{K-1} \mathbf{P}\left(y = k \,|\, x\right) \log\left(\frac{\mathbf{P}\left(y = k \,|\, x\right)}{\hat{\mathbf{P}}_\theta\left(y = k \,|\, x\right)}\right)$$

  - Average KL Divergence for all the data:

- Kullback–Leibler (KL) divergence for classification
  - For a **single** (x,y) data point

=2 Binary Classification

$$\mathbf{D}_{KL}\left(\mathbf{P}||\hat{\mathbf{P}}_\theta\right) = \sum_{k=0}^{K-1} \mathbf{P}\left(y = k \,|\, x\right) \log\left(\frac{\mathbf{P}\left(y = k \,|\, x\right)}{\hat{\mathbf{P}}_\theta\left(y = k \,|\, x\right)}\right)$$

  - Average KL Divergence for all the data:

$$\arg\min_\theta \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} \mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\frac{\mathbf{P}\left(y_i = k \,|\, x_i\right)}{\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)}\right)$$

Log(a/b) = Log(a) – Log(b)

Doesn't depend on θ: ~~$\mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\mathbf{P}\left(y_i = k \,|\, x_i\right)\right)$~~

$$- \mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

- Average **cross entropy loss**

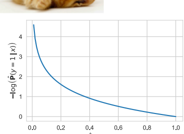
$$\arg\min_\theta \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} - \mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

- Average **cross entropy loss**

$$\arg\min_\theta \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} - \mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

Cute Cat Example

x =  , y = 1 "cute"



	Cute	Not Cute
Observed Probability	$\mathbf{P}\left(y = 1 \,|\, x\right)$ = 1.0	$\mathbf{P}\left(y = 0 \,|\, x\right)$ = 0.0
Predicted Probability	$\hat{\mathbf{P}}_\theta\left(y = 1 \,|\, x\right)$ = 0.8	$\hat{\mathbf{P}}_\theta\left(y = 0 \,|\, x\right)$ = 0.2
Cross Ent. $-\mathbf{P} \log \hat{\mathbf{P}}_\theta$	-1.0 Log(0.8) ≈ 0.22	-0.0 Log(0.2) = 0.0

Also called the log loss because it is the log of the predicted probability for the true class

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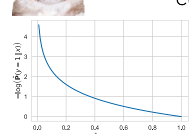
- Average **cross entropy loss**

$$\arg\min_\theta \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} - \mathbf{P}\left(y_i = k \,|\, x_i\right) \log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

Cute Cat Example

x =  , y = 0 "not cute"



	Cute	Not Cute
Observed Probability	$\mathbf{P}\left(y = 1 \,|\, x\right)$ = 0.0	$\mathbf{P}\left(y = 0 \,|\, x\right)$ = 1.0
Predicted Probability	$\hat{\mathbf{P}}_\theta\left(y = 1 \,|\, x\right)$ = 0.7	$\hat{\mathbf{P}}_\theta\left(y = 0 \,|\, x\right)$ = 0.3
Cross Ent. $-\mathbf{P} \log \hat{\mathbf{P}}_\theta$	-0.0 Log(0.7) = 0.0	-1.0 Log(0.3) ≈ 1.20

Also called the log loss because it is the log of the predicted probability for the true class

➢ Average **cross entropy loss**

$$\arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} -\mathbf{P}\left(y_i = k \,|\, x_i\right)\log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

➢ Computing the more general version for (x_i, y_i)

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}\mathbf{P}\left(y_i = 0 \,|\, x_i\right)\log\left(\hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right)\right) + \\ \quad \mathbf{P}\left(y_i = 1 \,|\, x_i\right)\log\left(\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right)\right)\end{array}\right)$$

$\mathbf{P}\left(y_i = 1 \,|\, x_i\right) = y_i$	$\hat{\mathbf{P}}_\theta\left(y_i = 1 \,|\, x_i\right) = \sigma(\phi(x_i)^T\theta)$
$\mathbf{P}\left(y_i = 0 \,|\, x_i\right) = (1 - y_i)$	$\hat{\mathbf{P}}_\theta\left(y_i = 0 \,|\, x_i\right) = 1 - \sigma(\phi(x_i)^T\theta)$

➢ Average **cross entropy loss**

$$\arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} -\mathbf{P}\left(y_i = k \,|\, x_i\right)\log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

➢ Computing the more general version for (x_i, y_i)

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(\begin{array}{l}(1 - y_i)\log\left(1 - \sigma(\phi(x_i)^T\theta)\right) + \\ \quad y_i \quad \log\left(\ \sigma(\phi(x_i)^T\theta)\ \right)\end{array}\right)$$

Rewriting on one line:

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\log\left(\sigma\left(\phi(x_i)^T\theta\right)\right) + (1 - y_i)\log\left(1 - \sigma\left(\phi(x_i)^T\theta\right)\right)\right)$$

➢ Average **cross entropy loss**

$$\arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}\sum_{k=0}^{K-1} -\mathbf{P}\left(y_i = k \,|\, x_i\right)\log\left(\hat{\mathbf{P}}_\theta\left(y_i = k \,|\, x_i\right)\right)$$

Rewriting on one line:

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\log\left(\sigma\left(\phi(x_i)^T\theta\right)\right) + (1 - y_i)\log\left(1 - \sigma\left(\phi(x_i)^T\theta\right)\right)\right)$$

After much algebra (see last lecture) we obtain:

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

The Loss for Logistic Regression

➢ Average **cross entropy** (simplified):

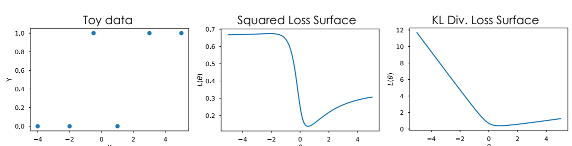
$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

- ➢ Equivalent to (derived from) **minimizing the KL divergence**
- ➢ Also equivalent to **maximizing the log-likelihood of the data** … (not covered in Data100 this semester)

Is this loss function reasonable?

Convexity Using Pictures

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

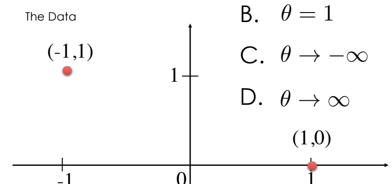


What is the value of θ?

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

Assume: $\phi(x) = x$

The Data

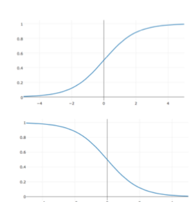


A. $\theta = -1$

B. $\theta = 1$

C. $\theta \to -\infty$

D. $\theta \to \infty$



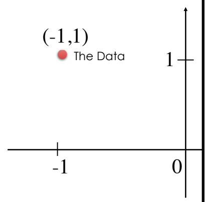
What is the value of θ?

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

Assume: $\phi(x) = x$

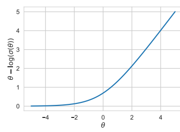
For the point (-1,1): $y_i\phi(x_i)^T = -1$

$-\phi(x_i)^T = 1$



Objective:

$\theta - \log\left(\sigma\left(\theta\right)\right)$



$\theta \to -\infty$

What is the value of θ?

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

Assume: $\phi(x) = x$

For the point (-1,1): $\theta - \log\left(\sigma\left(\theta\right)\right)$

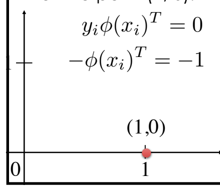
$\theta \to -\infty$

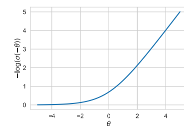
For the point (1, 0):

$y_i\phi(x_i)^T = 0$

$-\phi(x_i)^T = -1$

$0 - \log(\sigma(-\theta))$





$\theta \to -\infty$

What is the value of θ?

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

Assume: $\phi(x) = x$

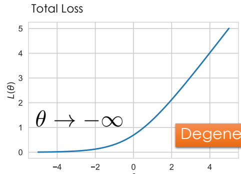
For the point (-1,1): $\theta - \log\left(\sigma\left(\theta\right)\right)$

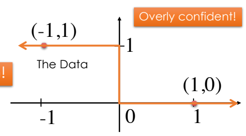
$\theta \to -\infty$

For the point (1, 0): $0 - \log(\sigma(-\theta))$

$\theta \to -\infty$

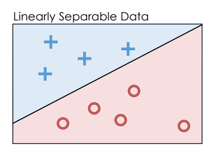
Total Loss

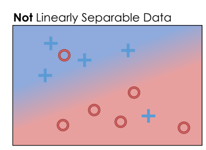


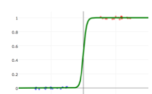


Linearly Separable Data

- A classification dataset is said to be linearly separable if there exists a hyperplane that separates the two classes.
- If data is linearly separable, logistic regression requires regularization







Weights go to infinity!

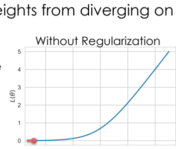
Solution?

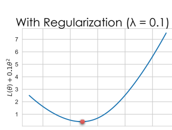
Adding Regularization to Logistic Regression

$$\arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right) + \lambda\sum_{j=1}^{d}\theta_j^2$$

- Prevents weights from diverging on linearly separable data

Earlier Example





Minimize the Loss

Logistic Loss Function

- Average KL divergence (simplified)

$$\arg\min_\theta -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

- Take Derivative:

$$\nabla_\theta \mathbf{L}(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\nabla_\theta y_i\phi(x_i)^T\theta + \nabla_\theta \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \nabla_\theta \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)$$

- Average KL divergence (simplified)

$$\arg\min_\theta -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

- Take Derivative:

$$\nabla_\theta \mathbf{L}(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\nabla_\theta y_i\phi(x_i)^T\theta + \nabla_\theta \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \nabla_\theta \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \frac{1}{\sigma\left(-\phi(x_i)^T\theta\right)}\nabla_\theta\sigma\left(-\phi(x_i)^T\theta\right)$$

- Take Derivative:

$$\nabla_\theta \mathbf{L}(\theta) = -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \frac{1}{\sigma\left(-\phi(x_i)^T\theta\right)}\nabla_\theta\sigma\left(-\phi(x_i)^T\theta\right)$$

Useful Identity

$$\frac{\partial}{\partial t}\sigma(t) = \frac{\partial}{\partial t}\frac{1}{1+e^{-t}} \overset{\text{Chain Rule}}{=} \frac{-1}{(1+e^{-t})^2}\frac{\partial}{\partial t}\left(1+e^{-t}\right)$$

$$\overset{\text{Chain Rule}}{=} \frac{e^{-t}}{(1+e^{-t})^2} \overset{\text{Alg.}}{=} \left(\frac{1}{1+e^{-t}}\right)\left(\frac{e^{-t}}{1+e^{-t}}\right)$$

$$\overset{\text{Alg.}}{=} \left(\frac{1}{1+e^{-t}}\right)\left(\frac{1}{e^{t}+1}\right) \overset{\text{Defn. of }\sigma}{=} \sigma(t)\sigma(-t)$$

- Take Derivative:

$$\nabla_\theta \mathbf{L}(\theta) = -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \frac{1}{\sigma\left(-\phi(x_i)^T\theta\right)}\nabla_\theta\sigma\left(-\phi(x_i)^T\theta\right)$$

Useful Identity $\frac{\partial}{\partial t}\sigma(t) = \sigma(t)\sigma(-t)$

$$= -\frac{1}{n}\sum_{i=1}^{n} y_i\phi(x_i) + \frac{\sigma\left(-\phi(x_i)^T\theta\right)}{\sigma\left(-\phi(x_i)^T\theta\right)}\sigma\left(\phi(x_i)^T\theta\right)\nabla_\theta(-\phi(x_i)^T\theta)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sigma\left(\phi(x_i)^T\theta\right)\right)\phi(x_i)$$

Logistic Loss Function

- Average KL divergence (simplified)

$$\arg\min_\theta -\frac{1}{n}\sum_{i=1}^{n}\left(y_i\phi(x_i)^T\theta + \log\left(\sigma\left(-\phi(x_i)^T\theta\right)\right)\right)$$

- Take Derivative:

$$\nabla_\theta \mathbf{L}(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \sigma\left(\phi(x_i)^T\theta\right)\right)\phi(x_i)$$

- Set derivative = 0 and solve for θ
  - No general analytic solution
  - Solved using numeric methods

The Gradient Descent Algorithm

$\theta^{(0)} \leftarrow$ initial vector (random, zeros …)

For τ from 0 to convergence:

$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \rho(\tau)\left(\nabla_\theta \mathbf{L}(\theta)\Big|_{\theta=\theta^{(\tau)}}^{\text{Evaluated at}}\right)$$

- ***ρ(τ)*** is the step size (learning rate)
  - typically $1/\tau$
- Converges when gradient is ≈ 0 (or we run out of patience)

Gradient Descent for Logistic Regression

Logistic Regression

$\theta^{(0)} \leftarrow$ initial vector (random, zeros …)

For τ from 0 to convergence:

$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \rho(\tau)\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sigma\left(\phi(x_i)^T\theta^{(\tau)}\right) - y_i\right)\phi(x_i)\right)$$

- ***ρ(τ)*** is the step size (learning rate)
  - typically $1/\tau$
- Converges when gradient is ≈ 0 (or we run out of patience)

Stochastic Gradient Descent

- For many learning problems the gradient is a sum:

$$\nabla_\theta \mathbf{L}(\theta) = \frac{1}{n}\sum_{i=1}^{n}\left(\sigma\left(\phi(x_i)^T\theta\right) - y_i\right)\phi(x_i)$$

  - For large n this can be costly
- What if we approximated the gradient by looking at a few random points:

$$\nabla_\theta \mathbf{L}(\theta) \approx \frac{1}{|\mathcal{B}|}\sum_{i\in\mathcal{B}}\left(\sigma\left(\phi(x_i)^T\theta\right) - y_i\right)\phi(x_i)$$

- What if we approximated the gradient by looking at a few random points:

$$\nabla_\theta \mathbf{L}(\theta) \approx \frac{1}{|\mathcal{B}|}\sum_{i\in\mathcal{B}}\left(\sigma\left(\phi(x_i)^T\theta\right) - y_i\right)\phi(x_i)$$

Batch Size

Random sample of records

- This is a reasonable estimator for the gradient
  - Unbiased …
- Often batch size is one! (why is this helpful)
  - Fast to compute!
- A key ingredient in the recent success of deep learning

Stochastic Gradient Descent

$\theta^{(0)} \leftarrow$ initial vector (random, zeros …)

For τ from 0 to convergence:

$$\mathcal{B} \sim \text{ Random subset of indices}$$

$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \rho(\tau)\left(\frac{1}{|\mathcal{B}|}\sum_{i\in\mathcal{B}}\nabla_\theta \mathbf{L}_i(\theta)\Big|_{\theta=\theta^{(\tau)}}\right)$$

Decomposable Loss

$$\mathbf{L}(\theta) = \sum_{i=1}^{n}\mathbf{L}_i(\theta) = \sum_{i=1}^{n}\mathbf{L}(\theta, x_i, y_i)$$

Loss can be written as a sum of the loss on each record.

Gradient Descent

$\theta^{(0)} \leftarrow$ initial vector (random, zeros …)

For τ from 0 to convergence:

$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \rho(\tau)\left(\frac{1}{n}\sum_{i=1}^{n}\nabla_\theta \mathbf{L}_i(\theta)\Big|_{\theta=\theta^{(\tau)}}\right)$$

Stochastic Gradient Descent

$\theta^{(0)} \leftarrow$ initial vector (random, zeros …)

Very Similar Algorithms

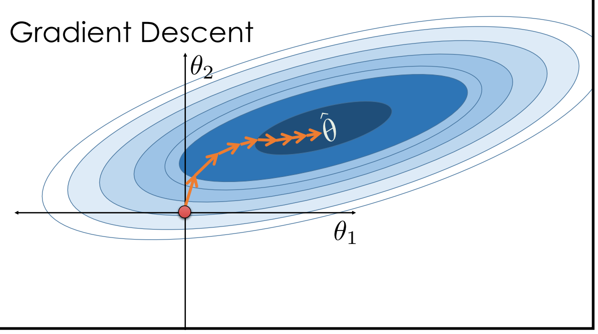
For τ from 0 to convergence:

$$\mathcal{B} \sim \text{ Random subset of indices}$$

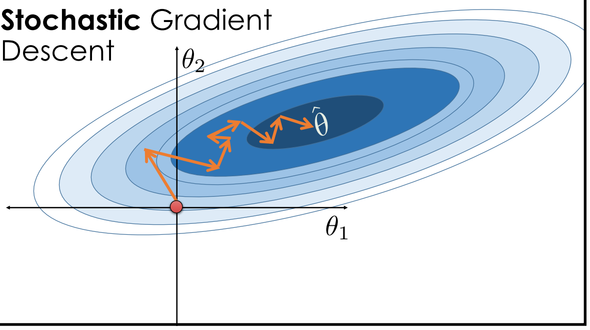
$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \rho(\tau)\left(\frac{1}{|\mathcal{B}|}\sum_{i\in\mathcal{B}}\nabla_\theta \mathbf{L}_i(\theta)\Big|_{\theta=\theta^{(\tau)}}\right)$$

Assuming Decomposable Loss Functions

Gradient Descent



Stochastic Gradient Descent



Logistic Regression in Scikit Learn

```
from sklearn.linear_model import LogisticRegression

# By default SK learn adds regularization
# C = 1/lambda the inverse regularization parameter.
model = LogisticRegression(C=100.00)

# Train the model
model.fit(df[['feat1', 'feat2']], df['label'])

# Make Predictions
test_df['label'] = model.predict(test_df[['feat1', 'feat2']])
test_df['P(Y|X)'] = model.predict_proba(test_df[['feat1', 'feat2']])
```

Python Demo!

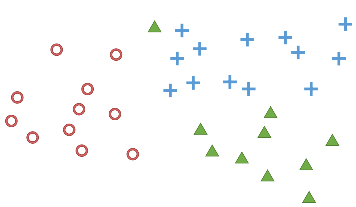
Attendance Quiz

http://bit.ly/ds100-sp18-sgd

1. Is the gradient of a **simple random sample** of the data an **unbiased estimate** of the gradient of the entire dataset? (T/F)
2. By decreasing the batch size we:
   1. **Increase** the **variance** (T/F)
   2. **Decrease** the **bias** (T/F)
   3. **Reduce** the **computational cost** of each iteration (T/F)

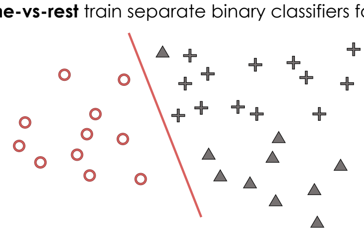
Multiclass (more than 2) Classification

- **One-vs-rest** train separate binary classifiers for each class



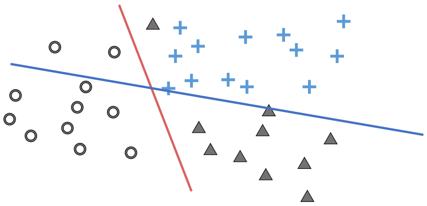
Multiclass (more than 2) Classification

- **One-vs-rest** train separate binary classifiers for each class



Multiclass (more than 2) Classification

- **One-vs-rest** train separate binary classifiers for each class



Multiclass (more than 2) Classification

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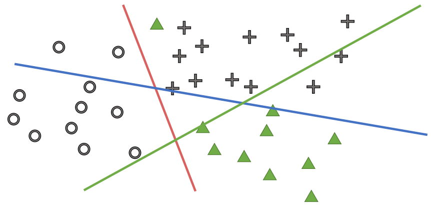

Multiclass (more than 2) Classification

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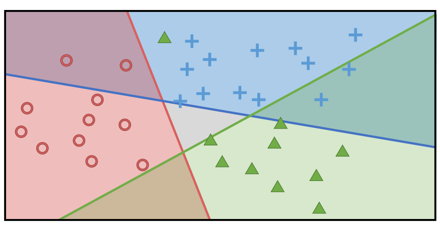
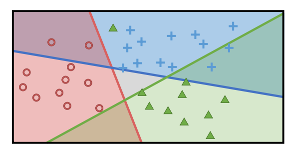


Multiclass (more than 2) Classification

- **One-vs-rest** train separate binary classifiers for each class
  - Class with highest confidence wins
  - Need to address class imbalance issue



- **Soft-Max** multiclass classification

- **Soft-Max** multiclass classification

$$\mathbf{P}\left(Y = j \,|\, x\right) = \frac{\exp\left(x^T \theta^{(j)}\right)}{\sum_{m=1}^{k} \exp\left(x^T \theta^{(m)}\right)}$$

- Separate $\theta^{(j)} \in \mathbb{R}^p$ for each class
- Trained using gradient descent methods
- Over parameterized. Why?
  - k sets of parameters one for each class
  - Only need K-1 parameters

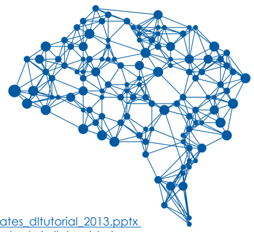
$$\mathbf{P}\left(y = k \,|\, x\right) = 1 - \sum_{j=1}^{K} \mathbf{P}\left(y = j \,|\, x\right)$$

  - Often use k parameters + regularization to address "redundancy".

Python Demo!

Deep Learning Overview

Bonus Material

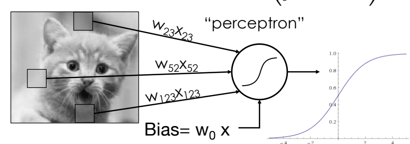


Borrowed from excellent talks by:
- **Adam Coates:** http://ai.stanford.edu/~acoates/coates_dltutorial_2013.pptx
- **Fei-Fei Li and Andrej Karpathy:** http://cs231n.stanford.edu/syllabus.html

Logistic Regression as a "Neuron"

- Consider the simple function family: $\sigma(u) = \frac{1}{1 + \exp(-u)}$

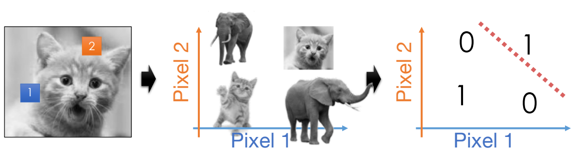
$$f_w(x) = \sigma\left(w^T x\right) = \sigma\left(\sum_{j=1}^{d} w_j x_j\right) = P(y = 1 \mid x)$$



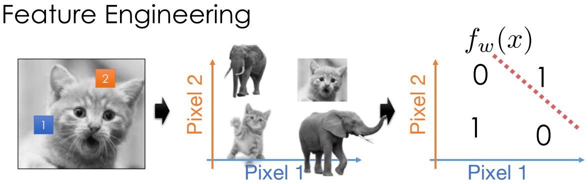
Neuron "fires" if weighted sum of input is greater than zero.

Logistic Regression: Strengths and Limitations

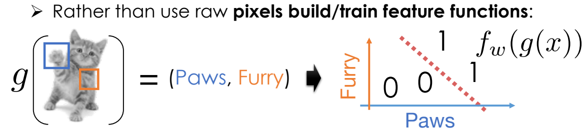
- Widely used machine learning technique
  - convex → efficient to learn
  - easy to interpret model weights
  - works well given good features
- Limitations:
  - Restricted to linear relationships → sensitive to choice of features



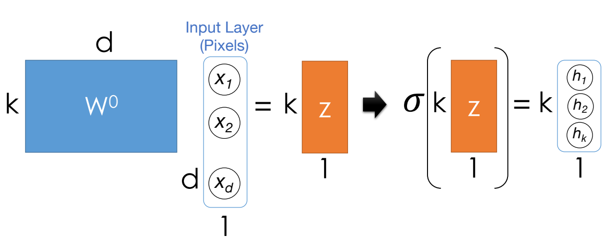
Feature Engineering



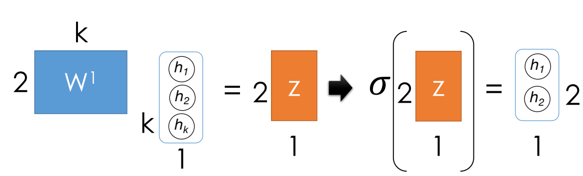
- Rather than use raw pixels **build/train feature functions**:



Composition Linear Models and Nonlinearities



Composition Linear Models and Nonlinearities

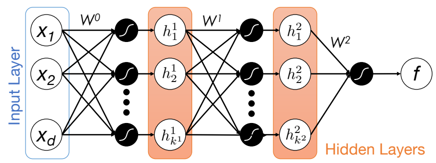


Neural Networks

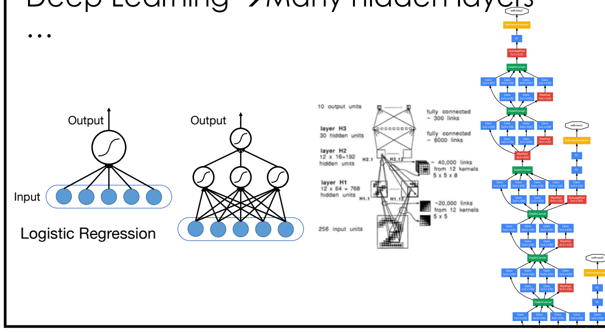
- Composing "perceptrons"

$$x \to \sigma(W^0 x) \to h^1 \to \sigma(W^1 h^1) \to h^2 \to \sigma(W^2 h^2) \to f$$

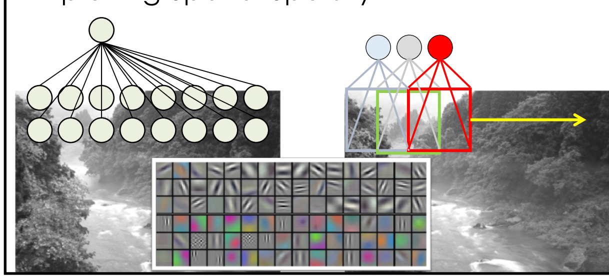
$$y = f_{W^0, W^1, W^2}(x) = \sigma\left(W^2\, \sigma\left(W^1\, \sigma\left(W^0 x\right)\right)\right)$$



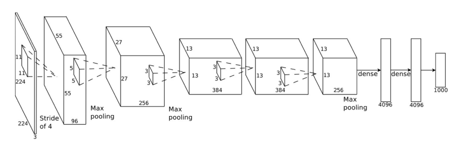
Deep Learning → Many hidden layers …



Convolutional Neural Networks: Exploiting Spatial Sparsity

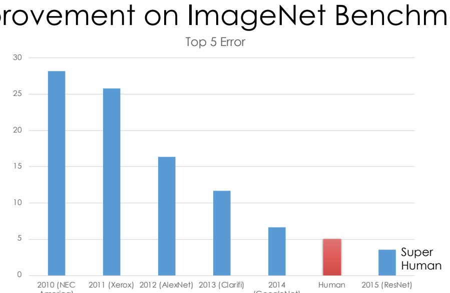


Example: AlexNet (Krizhevsky et al., NIPS 2012)

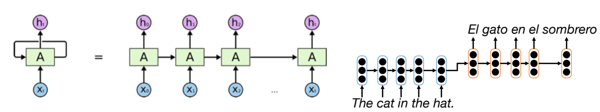


- Introduced in 2012, significantly outperformed state-of-the-art (top 5 error of 16% compared to runner-up with 26% error)

Improvement on ImageNet Benchmark

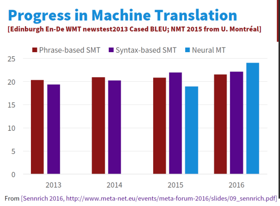


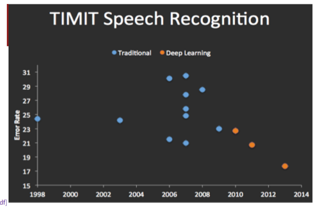
Recurrent Neural Networks: Modeling Sequence Structure



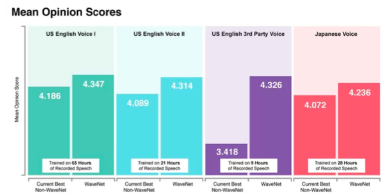
- input + previous output → new output
- State of the art in modeling sequential data
  - speech recognition and machine translation

Improvements in Machine Translation & Automatic Speech Recognition





State of the art in Text to Speech (TTS)



Interested in Deep Learning?

- RISE Lab Deep Learning Overview:
  - https://ucbrise.github.io/cs294-rise-fa16/deep_learning.html
- TensorFlow Python Tutorial
- Stanford CS231 Labs
  - http://cs231n.github.io/linear-classify/
  - http://cs231n.github.io/optimization-1/
  - http://cs231n.github.io/optimization-2/